The Born Approximation

In classical mechanics, the scattering cross section σ can be found by carefully determining the trajectories of the incoming "beam" particles and relating the scattering angle to the initial impact parameter. In quantum mechanics, there are no well-defined trajectories, so the procedure for calculating cross sections changes to one where the ration of incoming and outgoing fluxes to determine the probability that the scattering occurs. While exact analytic calculations of the cross section are often not possible, a useful approximation of the elastic scattering cross section is given by the first term in the Born series:

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4}\left|\int d^3r\, e^{-i\frac{\vec{q}\cdot\vec{r}}{\hbar}}\, V(\vec{r})\right|^2 \tag{1}$$

where $\vec{q}$ is the momentum transfer, $\hbar = \frac{h}{2\pi}$ is Planck's constant(h =6.63 $\times$ 10^{-34} joule-sec), m is the mass of the scattered particle, and $V(\vec{r})$ is the potential energy. The derivation of this result is beyond the present discussion, but we can use it to understand how nuclear form factors are determined.

- a) Use dimensional analysis to show that this expression has the same dimensions on both sides of the equation.
- b) Assume that $V(\vec{r}) = \frac{Zze^2}{4\pi\epsilon_0}\frac{e^{-\mu r}}{r}$. To calculate the Fourier transform of $V(r)$, begin by setting up the coordinates to be used in the integration. Since the coordinate directions inside the integral can be chosen arbitrarily, choose them so that $\vec{q}$ is in the positive z-direction, and then use spherical polar coordinates to do the integral. Do the integral over the azimuthal angle ϕ' first.
- c) Next, do the angular integration over θ'.
- d) Finally, do the integration over r by first replacing $\sin(qr)e^{-\mu r} = \Im(e^{(iq-\mu)r})$, integrating the complex exponential and then taking the imaginary part. d)Show that, in the limit $\mu \to 0$, the Born approximation reproduces the Rutherford scattering cross section.(Note that the magnitude of $|\vec{q}|$ is the same in both calculations.)

Born Again

The Rutherford scattering cross-section calculation assumes that the nucleus can be treated as a point charge. To go beyond this approximation, we need to calculate the electric potential energy between the nucleus and the scattered particle. Assume that the charge density of the nucleus is given by $\rho(\vec{r'})$, and the that scattered particle can be treated as a point charge.

- a) Make a sketch of the nucleus(a large sphere), where $\vec{r'}$ is the vector from the center of the nucleus to a particular location inside the nucleus. Draw in the particle to be scattered at a location outside the nucleus and call the vector that points from the center to the particle $\vec{r}$. Draw the vector that points between these two locations. What is it in terms of $\vec{r}$ and $\vec{r'}$?

- b) Since the nucleus is not a point, it cannot be treated as a point charge. We must imagine dividing it into tiny volumes, each of which is small enough to be thought of as pointlike, and then adding the contribution of each to the potential energy to get the total. Add the tiny volume d^3r' to the diagram. What is the charge inside this tiny volume?

- c) Calculate the contribution electric potential energy of the scattered particle due to the tiny charge located at $\vec{r'}$.

- d) By integrating over the entire nucleus, show that the potential energy of the scattered particle is given by

$$V(\vec{r}) = \frac{ze}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|} . (2)$$

- e) Write the expression for the Fourier transform of $V(\vec{r})$ in terms of two three dimensional integrals over $\vec{r}$ and $\vec{r'}$. By changing variables to $\vec{r} - \vec{r'}$ in one of the integrals, show that the Fourier transform of $V(\vec{r})$ is the product of the Fourier transforms of $\frac{ze}{4\pi\epsilon_0 r}$ and $\rho(\vec{r})$.

- f) Use this result to write the Born approximation to the cross section for a non-pointlike nucleus. Compare this to the expression in section 3-4 of Williams.