Normal Modes and Fourier Series

A string of length L and mass M is stretched tie onto two fixed supports, such that there is a tension T in the string. At $t = 0$, the string is displaced from equilibrium according to the equation $y(x, t = 0) = 0.25x(1 - x/L)$, and is initially released from rest so that $\frac{\partial y}{\partial t}(x, t)|_{t=0} = 0$.

- a) Sketch a graph of the initial displacement and velocity distribution on the string.
- b) The normal modes of the string are given by $y_n(x, t) = \sqrt{\frac{2}{L}} \sin(\frac{n\pi x}{L})(A_n \cos(\omega_n t) + B_n \sin(\omega_n t))$. Show that these functions solve the wave equation for the string with the appropriate boundary conditions for any values of A_n and B_n. What are the values of the ω_n required to do this?
- c) At any instant in time, the displacement and velocity of the string can be expressed as sum over the normal modes of the string and their time derivatives, repectively, through the proper choice of the arbitrary coefficients A_n and B_n. Therefore,

$$y(x, t) = \sum_{n=1}^{\infty} y_n(x, t). \tag{1}$$

  This remarkable fact is a result of the fact that the normal modes form a complete set of ortho-normal functions, which is the high-brow, mathematical way to say that

$$\frac{2}{L} \int_0^L \sin(\frac{n\pi x}{L}) \sin(\frac{n\pi x}{L}) = \delta_{nm} \tag{2}$$

  where δnm, known as the Kronecker delta symbol, is 1 if $n = m$ and zero otherwise. Show that this last statement is true.

- d) At $t = 0$, multiply both sides of Eq. 1 by $\sqrt{\frac{2}{L}} \sin(\frac{m\pi x}{L})$ and integrate from 0 to L to find an expression for A_m.
- e) At $t = 0$, multiply the time derivative of Eq. 1 by $\sqrt{\frac{2}{L}} \sin(\frac{m\pi x}{L})$ and integrate to find an expression for B_m.
- f) Substitute your results for A_m, B_m into Eq. 1, and obtain an expression for $y(x, t)$.
- g) Make a graph of $y(x, t)$ vs x for $t < 0.5L/c$, where c is the speed of the wave, in steps of $0.1L/c$.